Using the monotone measure sum to enrich the measurement of the interaction of multiple decision criteria

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Abstract. In the field of multicriteria decision analysis, the most distinctive characteristic of the monotone measure, or called the nonadditive measure, the fuzzy measure, the capacity, is that it can adequately and flexibly describe the interactions between the decision criteria. Traditionally, the interaction described with the monotone measure can be measured by the various kinds of the probabilistic interaction indices, among which the Shapley interaction index is the most famous and suitable one for the multicriteria decision making. Inspired by the marginal interaction of the multiple decision criteria, which is a core notion of the probabilistic interaction indices, we define the extremely positive and negative interaction cases of the multiple decision criteria, and find interestingly that the monotone measure sum (see Definitions 6 and 7) can be taken as an index to measure the kind and degree of the interaction of the multiple decision criteria. Further, we propose a new type of interaction index, i.e., the sum interaction index, and investigate its properties with respect to the additive, subadditive and superadditive monotone measures. Some comparison analyses of the sum interaction index with the Shapley interaction index are also given by several illustrative examples.

Keywords: Multicriteria decision analysis, monotone measure, shapley interaction index, sum interaction index

1. Introduction

In the framework of multicriteria decision analysis, a monotone measure [28, 36], or called capacity [11], nonadditive measure [12], fuzzy measure [38], is a normal monotone set function on the decision criteria set. By assessing the weight of any combination of the decision criteria, the monotone measure can describe flexibly the importance of each criterion as well as the interactions between them [20, 44]. The importance of decision criterion can be measured by its probabilistic value which can be interpreted as a mathematical expectation of the marginal contribution of the decision criterion [15, 20]. The simultaneous interaction among the decision criteria can be measured by the probabilistic interaction index [15, 20], which is a natural and reasonable extension of the probabilistic value [15]. The probabilistic interaction index of any subset can be similarly interpreted as a mathematical expectation of the marginal simultaneous interaction among the decision criteria of the subset [15, 25, 26]. The famous kinds of the probabilistic interaction indices usually include the Shapley interaction index [17], the Banzhaf interaction index [24], the chaining interaction index [24, 33] and the Möbius representation [9, 19]. Among them, the Shapley interaction index has some distinctive axiomatic characteristics [15], and hence has been universally accepted to represent the interaction phenomenon of the decision criteria in the literatures about the monotone measure and
Choquet integral based decision analysis (see, e.g., [2, 3, 5–7, 13, 14, 16–18, 20, 27, 43, 44]).

It is well known that one of the core compositions of the probabilistic interaction index is the marginal interaction of decision criteria, which is a natural extension of the marginal contribution of a single decision criterion. Since the monotonicity constraint of the monotone measure, the marginal contribution of any decision criterion is always nonnegative, and then the expectation of the marginal contribution, i.e., the probabilistic value, of any decision criterion is also nonnegative. However, the marginal interaction of decision criteria can be positive, negative or zero. As an expectation of the marginal interaction, the probabilistic interaction index can accordingly be positive, negative or zero. A positive (resp. negative) value of the interaction index means an extent of complementarity (resp. substitutivity) exists among the decision criteria; the zero of the interaction index means the decision criteria are independent to each other.

We can imagine two extreme cases of the interaction phenomena of all the decision criteria: the extreme complementarity and the extreme substitutivity. In the extreme complementarity case, the combination of all decision criteria, compared with the minor combination of some decision criteria as well as with the single decision criterion, is overwhelmingly important to the multicriteria decision making problem. This extreme complementarity can be described by the following monotone measure: the weight, i.e., the monotone measure value, of the universal set of decision criteria is 1, and the weights of all the other subsets of decision criteria are 0. In contrast, for the extreme substitutability case, any single decision criterion is completely sufficient for the decision making. That is, the importance of any single decision criterion is equal to the importance of any combination of decision criteria, even the importance of the universal set of all decision criteria. This extreme substitutivity can be described by the following monotone measure: only the weight of the empty set is 0, and the weights of all the nonempty subsets of decision criteria are 1. So, it is a natural requirement that, under the extreme complementarity and substitutivity cases, the interaction index value of the universal set should reach its maximum positive value and its minimum negative value, respectively. Unfortunately, the existing probabilistic interaction indices sometimes fail to satisfy this requirement.

Since the monotone measure is a normal monotone set function, one can easily figure out that the sum of the monotone measure values of all the subsets can be applied to represent and distinguish these two extreme interaction cases. Furthermore, the additive monotone measure, i.e., the classical probability measure, means that all the decision criteria are independent and in this situation the value of the interaction index of any combination of the decision criteria should preferably be zero. Inspired by these facts, in this paper, we construct a monotone measure sum based index to measure the interaction phenomenon of the multiple decision criteria. And we also verify that the sum interaction index satisfies some intuitive properties and can be taken as a feasible and alternative tool to represent the interaction among the multiple decision criteria.

This paper is organized as follows. After the introduction, we present some knowledge about the monotone measure and the probabilistic interaction indices in Section 2. In Section 3, we propose the monotone measure sum based interaction index and discuss the basic properties of this index. Finally, we conclude the paper and discuss future work in Section 4.

For convenience, let \( N = \{1, 2, \ldots, n\}, n \geq 2 \), be the decision criteria set, \( P(N) \) be the power set of \( N \).

### 2. Preliminaries

**Definition 1.** (See [11, 12, 20, 36, 38, 44]). A monotone measure on \( N \) is a set function \( \mu : P(N) \rightarrow [0, 1] \) such that

(i) \( \mu(\emptyset) = 0, \mu(N) = 1 \);

(ii) for \( \forall A, B \subseteq N, A \subseteq B \) implies \( \mu(A) \leq \mu(B) \).

Furthermore, a monotone measure \( \mu \) on \( N \) is said to be [20, 40, 42], additive (resp. superadditive, strict superadditive, subadditive, and strict subadditive), if \( \mu(A \cup B) = (\text{resp. } \geq, >, \leq \text{ and } <) \mu(A) + \mu(B) \) for \( \forall A, B \subseteq N \) and \( A \cap B = \emptyset \); to be symmetric or cardinality based if \( \mu(A) \) depends only on \( |A| \) for any \( A \subseteq N \), where \( |A| \) is the cardinality of set \( A \).

One can see that a monotone measure is a normal monotone set function which vanishes at the empty set [25]. In the framework of the multicriteria decision analysis, the number \( \mu(A) \) can be interpreted as the importance of the subset \( A \subseteq N \), and the monotonicity with respect to set inclusion (condition (ii) in Definition 1) means that the importance of a subset of criteria cannot decrease when new criteria are added to it [20]. This monotonicity constraint enables the monotone measure to flexibly represent the vari-
ous kinds of interactions among the decision criteria, ranging from substitutivity (negative interaction) to complementarity (positive interaction) [16, 25].

Generally speaking, the additive monotone measure means that the decision criteria are all independent to each other, i.e., the interaction of arbitrary group of decision criteria is zero. The strict superadditive (resp. strict subadditive) monotone measure means that the decision criteria are all mutually complementary (resp. substitutive), i.e., the interaction of arbitrary group of decision criteria is always positive (resp. negative). The interaction of arbitrary group of decision criteria associated with a superadditive (resp. subadditive) monotone measure means that every decision criterion plays a same role in the decision making process and then has the same overall importance.

When using a monotone measure to model the importance of the subsets of decision criteria, a suitable aggregation function is the Choquet integral [6, 20].

**Definition 2.** (See [11, 16, 17, 40, 42, 46]). Let $x$ be a real-valued function on $N$, $x := (x_1, x_2, \ldots , x_n)$, the Choquet integral of $x$ with respect to a monotone measure $\mu$ on $N$ is defined as

$$C_\mu(x) = \sum_{i=1}^{n} [x(i) - x(i-1)] \mu(N(i))$$

or equally by

$$C_\mu(x) = \sum_{i=1}^{n} [\mu(N(i)) - \mu(N(i+1))]x(i)$$

where the parentheses used for indices represent a permutation on $N$ such that $x(1) \leq \cdots \leq x(n)$, $x(0) = 0$, $N(i) = \{i\}, \ldots , \{n\}$, and $N(n+1) = \emptyset$.

The Choquet integral has some good aggregation properties, such as idempotence, compensativeness and comonotonic additivity [30, 42]. The Choquet integral has been widely used as an aggregation tool in many applications [8, 21], such as decision making [10, 16, 17, 29, 35], computation of the number of citations [39], industrial performance improvement [7], information classification and fusion [1, 34, 45, 46].

In the context of the monotone measure and Choquet integral based multicriteria decision analysis, the overall importance of a decision criterion is measured by the probabilistic value and the simultaneous interaction among multiple decision criteria is usually measured by the probabilistic interaction index.

**Definition 3.** (See [15, 41]). Let $\mu$ be a monotone measure on $N$, the probabilistic value of a criterion $i \in N$ with respect to $\mu$ is defined as

$$I^\mu_i(i) = \sum_{B \subseteq N \setminus \{i\}} p^B_i(N)[\mu(B \cup \{i\}) - \mu(B)],$$

where, for any $i \in N$, the family of coefficients $\{p^B_i(N)\}_{B \subseteq N \setminus \{i\}}$ forms a probability distribution on $P(N \setminus \{i\})$. Additionally, for any $i \in N$ and $B \subseteq N \setminus \{i\}$, if the coefficient $p^B_i(N)$ depends only on $|\{i\}|$, $|B|$ and $|N|$, i.e., the cardinalities of the subsets $\{i\}$, $B$ and $N$, then the probabilistic value is called as a cardinal probabilistic value or a semivalue [28].

The two most famous probabilistic values, the Shapley value [37] and the Banzhaf value [4], are both the cardinal probabilistic values. Let $\mu$ be a monotone measure on $N$, the Shapley value of a criterion $i \in N$ with respect to $\mu$ is defined as

$$I^\mu_{Sh}(i) = \sum_{B \subseteq N \setminus \{i\}} \frac{|B|!|N|! - |B|!}{|N|!} [\mu(B \cup \{i\}) - \mu(B)],$$

the Banzhaf value of a criterion $i \in N$ with respect to $\mu$ is defined as

$$I^\mu_{Bn}(i) = \sum_{B \subseteq N \setminus \{i\}} \frac{1}{2(|N| - 1)} [\mu(B \cup \{i\}) - \mu(B)].$$

**Definition 4.** (See [15, 24]). Let $\mu$ be a monotone measure on $N$, the probabilistic interaction index of any subset $A \subseteq N$ with respect to $\mu$ is defined as

$$I^\mu_p(A) = \sum_{B \subseteq N \setminus A} p^A_B(N) \left( \sum_{C \subseteq A} (-1)^{|A \setminus C|} \mu(C \cup B) \right),$$

where, for any $A \subseteq N$, the family of coefficients $\{p^A_B(N)\}_{B \subseteq N \setminus A}$ forms a probability distribution on $P(N \setminus A)$. Additionally, for any $A \subseteq N$ and $B \subseteq N \setminus A$, if the coefficient $p^A_B(N)$ depends only on $|A|$, $|B|$ and $|N|$, the probabilistic interaction index is called as the cardinal probabilistic interaction index.

By comparing Definitions 3 and 4, one can see that the probabilistic interaction index is a natural generalization of the probabilistic value. In Definition 3, $\mu(B \cup \{i\}) - \mu(B)$ can be thought of as the marginal contribution of the criterion $i$ to the subset $B$ not containing it [20], then the probabilistic value $I^\mu_p(i)$ can be interpreted as the mathematical expectation of the marginal contribution of criterion $i$ [15]. So,
the larger the probabilistic value $I^\mu_p((i))$ is, the more important the criterion $i$ is to the decision making. From Definition 4, for two criteria $i$ and $j$, we have the probabilistic interaction index

\[ I^\mu_p((i, j)) = \sum_{B \subseteq N \setminus \{i, j\}} p_B^\mu((i))(\mu(B \cup \{i, j\})) - \mu(B) - \mu(B \cup \{i\}) + \mu(B \cup \{j\}). \]

when $B = \emptyset$, we have $\mu(B \cup \{i, j\}) - \mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B) = \mu(i, j) - \mu(i) - \mu(j)$. Then, the inequality $\mu((i, j)) - \mu((i)) - \mu((j)) > 0$ seems to model a positive interaction or complementary effect between criteria $i$ and $j$ [15, 20], the inequality $\mu((i, j)) - \mu((i)) - \mu((j)) < 0$ suggests that criteria $i$ and $j$ interact in a negative or substitutive way [15, 20], the equation $\mu((i, j)) - \mu((i)) - \mu((j)) = 0$ means that the criteria $i$ and $j$ do not interact, i.e., they are independent [15, 20]. In general, for $\forall B \subseteq N \setminus \{i, j\}$, the expression $\mu(B \cup \{i, j\}) - \mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B)$ can be considered as the marginal interaction between the criteria $i$ and $j$ with the presence of $B$ [15, 20]. Therefore the quantity $I^\mu_p((i, j))$ can be interpreted as the mathematical expectation of the marginal interaction between the criteria $i$ and $j$ [15, 20]. Similarly, the quantity $I^\mu_p(A)$, for $\forall A \subseteq N$ and $|A| > 2$, can be considered as the mathematical expectation of marginal simultaneous interaction among the criteria of the subset $A$ [15, 20], and can be used to measure the simultaneous interaction among the decision criteria [15].

The following definition gives four well-known probabilistic interaction indices.

**Definition 5.** (See [9, 15, 19, 22, 23, 33]). Let $\mu$ be a monotone measure on $N$, the Shapley interaction index of any subset $A \subseteq N$ with respect to $\mu$ is defined as

\[ I^\mu_{ch}(A) = \sum_{B \subseteq N \setminus A \cup \{A\}} \frac{1}{(|N| - |A| + 1)(|N| - |A| + |B|)} \left( |N| - |A| \right)^{-1} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right), \]

the Banzhaf interaction index of any subset $A \subseteq N$ with respect to $\mu$ is defined as

\[ I^\mu_{ba}(A) = \sum_{B \subseteq N \setminus A \cup \{A\}} \frac{1}{2(|N| - |A|)} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right). \]

the chaining interaction index of any subset $A \subseteq N$, $A \neq \emptyset$, with respect to $\mu$ is defined as

\[ I^\mu_{ch}(A) = \sum_{B \subseteq N \setminus A} \frac{|A|}{|A| + |B|} \left( \frac{|N|}{|A| + |B|} \right)^{-1} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right), \]

the Möbius representation of any subset $A \subseteq N$ with respect to $\mu$ is defined as

\[ m_\mu(A) = \sum_{B \subseteq A} (-1)^{|A\setminus B|} \mu(B). \]

The Shapley interaction index and the chaining interaction index both extend the Shapley value [15, 24, 37] while the Banzhaf interaction index just extends the Banzhaf value [4]. The above four indices are all the cardinal probabilistic interaction indices. The coefficients of the former three types of interaction indices obviously depend only on $|A|$, $|B|$ and $|N|$, the coefficients of Möbius representation are defined by $p_N^\mu(N) = 1$ if $|B| = 0$ (i.e., $B = \emptyset$) and 0 otherwise [15, 22, 24]. The Möbius representation is also called as internal interaction index [15] since it only cares about the interaction within the subset and does not take account of the nonempty presence. The Möbius representation plays a great role in the monotone measure identification processes since it provides an efficient and alternative way to represent the monotone measure, the Choquet integral and the Shapley interaction index. It should be pointed out that, among the four cardinal probabilistic interaction indices, the Shapley interaction index has some distinctive axiomatic properties, such as linearity axiom, dummy axiom, symmetry axiom, efficiency axiom and recursive axiom [15], and hence the Shapley interaction index is the most suitable one to measure the simultaneous interaction among the multicriteria decision criteria [15, 23].

3. The monotone measure sum based interaction index

For the sake of clarity, we consider the simplest situation that the decision criteria set only consists of two criteria, $N = \{1, 2\}$, $\mu$ is a monotone measure on $N$. Then according to the analysis of the marginal interaction in Section 2, the inequality $\mu(N) > \mu(\{1\}) + \mu(\{2\})$ means the interaction...
between the two criteria 1 and 2 is positive. Further, it is natural and reasonable to believe that the difference between left side and right side of the above inequality, \( \mu(N) - \mu(\{1\}) - \mu(\{2\}) \), can reflect the degree of the positive interaction between the criteria 1 and 2. The larger the difference is, the higher the degree of the positive interaction should be. From Definition 1, we know that \( 0 = \mu(\emptyset) \leq \mu(\{1\}), \mu(\{2\}) \leq \mu(N) = 1 \). Hence, the largest difference between the two sides is \( \mu(N) - \mu(\{1\}) - \mu(\{2\}) = 1 \), i.e., \( \mu(N) = 1 \) and \( \mu(\{1\}) = \mu(\{2\}) = 0 \), which means that the extremely positive interaction exists between the criteria 1 and 2. In contrast, \( \mu(N) < \mu(\{1\}) + \mu(\{2\}) \) means the interaction between the two criteria is negative. Then \( \mu(N) - \mu(\{1\}) - \mu(\{2\}) = -1 \), i.e., \( \mu(N) = \mu(\{1\}) = \mu(\{2\}) = 1 \), means that the extremely negative interaction exists between the decision criteria 1 and 2. Furthermore, \( \mu(N) - \mu(\{1\}) - \mu(\{2\}) = 0 \), i.e., \( \mu(N) = \mu(\{1\}) + \mu(\{2\}) \), means that the decision criteria 1 and 2 are independent to each other, and in this situation the monotone measure is additive.

The situation is slightly more complex when the decision criteria set consists of three criteria, \( N = \{1, 2, 3\} \). The extremely positive interaction of the three decision criteria can be described by the following monotone measure:

\[
\mu(N) = 1, \quad \text{and} \quad \mu(\emptyset) = \mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{1, 2\}) = \mu(\{1, 3\}) = \mu(\{2, 3\}) = \mu(N) = 1, \quad \text{and} \quad \mu(\emptyset) = 0.
\]

The extremely negative interaction of the three decision criteria can be described by the following monotone measure:

\[
\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{1, 2\}) = \mu(\{1, 3\}) = \mu(\{2, 3\}) = \mu(N) = 1, \quad \text{and} \quad \mu(\emptyset) = 0.
\]

The zero interaction case, i.e., the three decision criteria are independent to each other, can be described by an additive monotone measure on \( N = \{1, 2, 3\} \).

Generally, for the decision criteria, \( N = \{1, 2, \ldots, n\} \), \( n \geq 2 \), the extremely positive interaction existing among all the decision criteria can be described by the monotone measure \( \mu_{ep} \) as:

\[
\mu_{ep}(N) = 1 \quad \text{and} \quad \mu_{ep}(\emptyset) = 0 \quad \text{for} \quad \forall A \subseteq N.
\]

The extremely negative interaction existing among all the decision criteria can be described by the monotone measure \( \mu_{en} \) as:

\[
\mu_{en}(A) = 1 \quad \text{for} \quad \forall A \subseteq N \quad \text{and} \quad A \neq \emptyset, \quad \text{and} \quad \mu_{en}(\emptyset) = 0.
\]

The zero interaction case, i.e., the decision criteria are all independent to each other, can be described by an additive monotone measure on \( N \).

The intuitive explanations of the above extremely positive and negative interaction cases can be given as follows. In the extremely positive case, all the decision criteria are totally complementary with each other, and the universal set of the decision criteria is overwhelmingly more important than any proper subset and single decision criteria. A desired decision alternative must satisfy all the decision criteria and cannot fail in any decision criterion. In contrast, for the extremely negative interaction case, all the decision criteria are totally substitutive with each other, the importance of the single decision criteria is equal to that of any nonempty subset of the decision criteria, and the decision alternative that satisfies any single decision criterion can be chosen as the desired one.

Now, our task is to find a simple and suitable index to distinguish the extremely positive and negative interaction cases, or equally to distinguish the monotone measures \( \mu_{ep} \) and \( \mu_{en} \). As mentioned in the introduction, since the monotone measure is a normal monotone set function (see Definition 1), these two monotone measures \( \mu_{ep} \) and \( \mu_{en} \) can be completely identified by the sum of monotone measure values of all the subsets, denoted as \( \sum_{A \subseteq N} \mu(A) \).

**Proposition 1.** Let \( \mu \) be a monotone measure on \( N \), \( \sum_{A \subseteq N} \mu(A) \) be the sum of monotone measure values of all the subsets, then \( \sum_{A \subseteq N} \mu(A) \) equals 1 if and only if the monotone measure \( \mu = \mu_{ep} \), and sum(\mu) equals \( 2^n - 1 \) if and only if the monotone measure \( \mu = \mu_{en} \).

That is, \( \sum_{A \subseteq N} \mu(A) = 1 \Leftrightarrow \mu = \mu_{ep} \), and \( \sum_{A \subseteq N} \mu(A) = 2^n - 1 \Leftrightarrow \mu = \mu_{en} \).

**Proof.** It can be easily obtained from the fact that \( \mu(A) \geq \mu(\emptyset) = 0 \) and \( \mu(A) \leq \mu(N) = 1 \) for \( \forall A \subseteq N \).

The index \( \sum(\mu) \) can distinguish the monotone measures \( \mu_{ep} \) and \( \mu_{en} \), and hence can be adopted as an index to measure the extremely positive and negative interaction among all the decision criteria. It should be pointed out that there are many other feasible indices to distinguish the monotone measures \( \mu_{ep} \) and \( \mu_{en} \), such as \( k \sum_{A \subseteq N} \mu(A) \) and \( k(\mu(N) - \sum_{A \subseteq N} \mu(A)) \) when \( k \neq 0 \). The main reason to adopt the index \( \sum_{A \subseteq N} \mu(A) \) is because of its simple structure.

Here, we further take into account the additive monotone measures which describe the cases that all the decision criteria are mutually independent.

**Proposition 2.** Let \( \mu_a \) be an additive monotone measure on \( N \), then \( \sum_{A \subseteq N} \mu_a(A) = 2^n - 1 \).
Proof. We can first consider the sum of the monotone measure values of the subsets whose cardinalities are 1, i.e., the sum of $\mu_a(\{1\}), \mu_a(\{2\}), \ldots, \mu_a(\{n\})$. Since $\mu_a$ is an additive monotone measure on $N$, we get that $\sum_{i \in N} \mu_a(\{i\}) = \mu(N) = 1$. For the subsets with cardinality of 2, we have

$$\sum_{i,j \in N} \mu_a(\{i, j\}) = \sum_{i,j \in N} (\mu_a(\{i\}) + \mu_a(\{j\})) = \binom{n-1}{1} \sum_{i \in N} \mu_a(\{i\}) = \binom{n-1}{1} = \binom{n-1}{1}.$$

For the subsets with cardinality of 3, we have

$$\sum_{i,j,k \in N} \mu_a(\{i, j, k\}) = \sum_{i,j \in N} (\mu_a(\{i\}) + \mu_a(\{j\}) + \mu_a(\{k\})) = \binom{n-1}{2} \sum_{i \in N} \mu_a(\{i\}) = \binom{n-1}{2}.$$

Similarly, we can get that the sum of monotone measure values of the subsets with cardinality of $l$ is equal to $\binom{n-1}{l-1}$. Hence, we have

$$\text{sum}_N(\mu_a) = \sum_{i=1}^{n} \binom{n-1}{i-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}.$$

Unfortunately, the converse of the proposition 2 is not true. Let $\mu$ be a monotone measure on $N$, $|N| > 2$, if $\text{sum}_N(\mu)$ is equal to $2^{n-1}$, then this monotone measure $\mu$ is not necessarily additive.

Example 1. Let $N = \{1, 2, 3\}, \mu_1, \mu_2, \mu_3$ and $\mu_4$ be four monotone measures on $N$. Their measure values of any subset of $N$ are given in Table 1. It is obvious that $\mu_1, \mu_2, \mu_3$ and $\mu_4$ are not additive, but their sum are all equal to $4 = 2^2$. For the monotone measure $\mu_1$, the importance of any single decision criterion is zero, and the importance of any pair of criteria is 1. This means that there are some complementary ingredients between the decision criteria. On the other hand, the universal set of decision criteria and all the pairs of decision criteria have the same importance 1, which means that there are some substitutive ingredients among the decision criteria. The interaction situation of $\mu_2$ is similar to that of $\mu_1$. For the monotone measure $\mu_4$, the importance of the single decision criterion is equal to that of the pair of decision criteria, and this means there are some substitutive ingredients between the decision criteria. Furthermore, the importance of the pair decision criteria is only 0.5 while the importance of the set of the three decision criteria is 1, which means there are some complementary ingredients among them. The monotone measure $\mu_3$ has the similar interaction situation.

In the above example, the complementary and substitutive interactions relatively evenly coexist among the decision criteria for the four monotone measures. From a comprehensive perspective, there should be a trade-off of the two kinds of interactions in the final result that given by an interaction measure index. If the sum of a monotone measure on $N$ is equal to the sum of the additive monotone measure, $2^{n-1}$, we can say the two kinds of interactions associated with this monotone measure are relatively balanced, and the comprehensive interaction among all the decision criteria with respect to this monotone measure is approximately equivalent to that with respect to the additive monotone measure. So, it is relatively reasonable to assume that the comprehensive interaction among all the decision criteria is ‘zero’ if the sum of the monotone measure on $N$ is equal to $2^{n-1}$.

In summary, if the sum of the monotone measure on $N$ is equal to $2^n - 1$, $2^{n-1}$ and 1, then we say that the comprehensive interaction among all the decision criteria is extremely negative, zero and extremely positive, respectively. For convenience, we can normalize the sum of the monotone measure into the interval $[-1, 1]$ and take it as an interaction index of the universal set $N$.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>The monotone measures $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\mu_1(A)$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>0</td>
</tr>
<tr>
<td>${1}$</td>
<td>0.1</td>
</tr>
<tr>
<td>${2}$</td>
<td>0.1</td>
</tr>
<tr>
<td>${3}$</td>
<td>0.1</td>
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<tr>
<td>${1,2}$</td>
<td>0.9</td>
</tr>
<tr>
<td>${1,3}$</td>
<td>0.9</td>
</tr>
<tr>
<td>${2,3}$</td>
<td>1</td>
</tr>
<tr>
<td>${1,2,3}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Definition 6. Let $N = \{1, 2, \ldots, n\}, n \geq 2$, and $\mu$ be a monotone measure on $N$, the monotone measure sum based interaction index of $N$, or called the sum interaction index of $N$ for short, is defined as

$$I_{\text{sum}}^\mu(N) = \frac{2^n - 1 - \text{sum}_N(\mu)}{2^{n-1} - 1},$$

where $\text{sum}_N(\mu) = \sum_{A \subseteq N} \mu(A)$. 
If the sum interaction index $I_{\text{sum}}^\mu(N)$ is equal to $-1$, 0 and 1, then the corresponding comprehensive interaction among all the decision criteria of $N$ is said to be extremely negative, zero and extremely positive, respectively.

Since

$$I_{\text{sum}}^\mu(N) = \frac{2^{n-1} - \text{sum}_N(\mu)}{2^{n-1} - 1}$$

$$= \frac{2^{|N|-1} - \sum_{A \subseteq N} \mu(A)}{2^{|N|-1} - 1}$$

$$= \frac{2^{|N|-1} \mu(N) - \sum_{A \subseteq N} \mu(A)}{2^{|N|-1} \mu(N) - \mu(N)}$$

we can get the following definition of the sum interaction index of any subset $A \subseteq N$.

**Definition 7.** Let $\mu$ be a monotone measure on $N$, for $\forall A \subseteq N$, the sum interaction index of $A$ is defined as

$$I_{\text{sum}}^\mu(A) = \frac{2^{|A|-1} \mu(A) - \sum_{B \subseteq A} \mu(B)}{2^{|A|-1} \mu(A) - \mu(A)}$$

if $2^{|A|-1} \mu(A) - \mu(A) \neq 0$, and $I_{\text{sum}}^\mu(A) = 0$ otherwise, where $\sum_{B \subseteq A} \mu(B)$ is the cardinality of the subset $A$.

One can see that $I_{\text{sum}}^\mu(A) = 0$ if and only if $2^{|A|-1} \mu(A) - \sum_{B \subseteq A} \mu(B) = 0$.

**Proposition 3.** The sum interaction index of any singleton set as well as of the empty set equals 0, i.e., $I_{\text{sum}}^\mu(i) = 0$ for $\forall i \in N$, and $I_{\text{sum}}^\mu(\emptyset) = 0$.

**Proof.** It can be directly obtained from Definition 7.

The property shown in the proposition 3 is a main difference between the sum interaction index and the probabilistic interaction index. Most types of the probabilistic interaction indices of the singleton set, i.e., the probabilistic values, e.g., the Shapley value and Banzhaf value, can be positive or zero. Especially, in the multicriteria decision analysis context, the Shapley value can be taken as the overall importance of the decision criterion and hence is usually not equal to zero. From Definition 5, we can get that the Môbius representation of the empty set is always 0, the Shapely and Banzhaf interaction indices of the empty set are never zero. The Shapley interaction index of empty set can be interpreted as the expectation of the Choquet integral with a uniform distribution on the unit $[0, 1]$ [31], and can be used to measure the disjunctive or conjunctive trend (aggregates like a maximum or minimum operator) of the Choquet integral [31, 32]. As for the sum interaction index, since the interaction is essentially a relationship among multiple decision criteria, so it is acceptable that the sum interaction index value of the empty set is never zero. The Shapley interaction index values of every decision criterion and the empty set are all zero. From Table 3, we have all the Shapley interaction index values of any singleton set as well as of the empty set equals 0.

**Example 1 (continued).** The sum interaction indices of the four monotone measures are listed in Table 2. For the purpose of comparison, the Shapley interaction indices of the four monotone measures are given in Table 3.

One can see that, in Table 2, the sum interaction index values of every decision criterion and the empty set are all zero. From Table 3, we have all the Shapley interaction index values of the empty set are equal to 0.500, which means that the Choquet integrals with respect to the four monotone measures act more like a median operator (since the Shapley interaction index of the empty set is equivalent to the expectation of the Choquet integral [31, 32], and the expectation value 0 (resp. 0.5 and 1) of an aggregation function means it acts like a minimum (resp. median and maximum) operator). The Shapley values of every decision criterion with respect to the four monotone measures are all 1/3, which means that the three decision criteria have the same overall importance. Comparing Tables 2 and 3, one can see that there are some differences between the values of the two kinds of interaction indices.
Proposition 4. Let $\mu$ be a monotone measure on $N$, then $\mu$ is additive if and only if the sum interaction index of any subset of $N$ is equal to 0, i.e., $\mu$ is additive $\iff I_{\text{sum}}^\mu(A) = 0$ for $\forall A \subseteq N$.

Proof. From Definition 7, we can get that, for $\forall A \subseteq N$, $I_{\text{sum}}^\mu(A) = 0$ if and only if $2^{\lvert A \rvert - 1} \mu(A) - \sum_{B \subseteq A} \mu(B) = 0$. From the proof of the proposition 2, we can get that $\mu$ is additive $\implies 2^{\lvert A \rvert - 1} \mu(A) - \sum_{B \subseteq A} \mu(B) = 0$. Now, consider the necessary condition. When $\lvert A \rvert = 2$, we can assume $A = \{i, j\}$, then

$$2\mu(i, j) - \sum_{B \subseteq \{i, j\}} \mu(B) = 0$$

$\implies 2\mu(i, j) - \mu(i) - \mu(j) = 0$.

When $\lvert A \rvert = 3$, we can assume $A = \{i, j, k\}$, then

$$2^2 \mu(i, j, k) - \sum_{B \subseteq \{i, j, k\}} \mu(B) = 0$$

$\implies -\mu(i, j) - \mu(i, k) - \mu(j, k) - \mu(i) - \mu(j) - \mu(k) = 0$.

When $\lvert A \rvert = 4$, we have $\mu(i, j, k, l) = \mu(i, j) + \mu(i, k) + \mu(i, l) + \mu(j, k) + \mu(j, l) + \mu(k, l)$.

Assume that all the subsets with cardinality less than or equal to $k$ satisfy the additive condition, i.e., for $\forall A \subseteq N$ and $\lvert A \rvert \leq k$, we have $\mu(A) = \sum_{i \in A} \mu(i)$. Let us consider the subset $A$ with $\lvert A \rvert = k + 1$, we have $2^{\lvert A \rvert - 1} \mu(A) - \sum_{B \subseteq A} \mu(B) = 0 \iff 2^{\lvert A \rvert - 1} \mu(A) = \sum_{B \subseteq A} \mu(B)$. Since for $\forall i \in A$, there are $2^{\lvert A \rvert - 1}$ subsets containing the element $i$, we can have $2^{\lvert A \rvert - 1} \mu(A) = \sum_{i \in A} 2^{\lvert A \rvert - 1} \mu(i)$. That is, $\mu(A) = \sum_{i \in A} \mu(i)$ when $\lvert A \rvert = k + 1$. Hence, we have $2^{\lvert A \rvert - 1} \mu(A) - \sum_{B \subseteq A} \mu(B) = 0 \iff \mu$ is additive.

The Shapley interaction index also has a similar property, i.e., the dummy axiom [15, 23, 24]. Let $\mu$ be a monotone measure on $N$, a decision criterion $i \in N$ is said to be dummy [24] if $\mu(A \cup \{i\}) = \mu(A) + \mu(i)$ for any $A \subseteq N \setminus \{i\}$. The dummy axiom is that if $i \in N$ is a dummy, then $I_{\text{Sh}}^\mu(\{i\}) = \mu(i)$, $I_{\text{Sh}}^\mu(A \cup \{i\}) = 0$ for any $A \subseteq N \setminus \{i\}$ such that $A \neq \emptyset$. The dummy axiom means that the overall importance of the dummy criterion is equal to its monotone measure value and that the dummy criterion does not interact with any other subset [15, 23, 24]. That is, the Shapley interaction index, or the simultaneous interaction, of the subset that contains a dummy criterion will equal to zero. Since for the additive monotone measure, every decision criterion can be regarded as a dummy criterion, then we have that $\mu$ is additive $\iff I_{\text{Sh}}^\mu(A) = 0$ for $\forall A \subseteq N$, $\lvert A \rvert \geq 2$.

It should be mentioned that converse of this statement is not true. Furthermore, the sum interaction index of the subset that containing the dummy criteria is not always zero.

Example 2. Let $N = \{1, 2, 3\}$, $\mu_5$ is a monotone measure on $N$ and is given in Table 4. Its Shapley interaction index and sum interaction index are also listed in Table 4. One can see that the criterion 3 is a dummy criterion. According to the dummy axiom, $I_{\text{Sh}}^\mu_5(\{3\}) = \mu_5(\{3\}) = 0.4$ and $I_{\text{Sh}}^\mu_5(\{1, 3\}) = I_{\text{Sh}}^\mu_5(\{2, 3\}) = I_{\text{Sh}}^\mu_5(\{1, 2, 3\}) = 0$. That is, the simultaneous interactions between criteria 1 and 3, between criteria 2 and 3, among criteria 1, 2 and 3 are all zero. In contrast, the sum interaction index $I_{\text{sum}}^\mu_5(\{1, 2, 3\}) = 0.2$ shows that the sum interaction among the three criteria is 0.2.

Proposition 5. Let $\mu$ be a monotone measure on $N$, if $\mu$ is subadditive then the sum interaction index of any subset of $N$ is nonpositive, and if $\mu$ is strict subadditive then the sum interaction index of the nonempty nonsingleton subset of $N$ is negative, i.e., $\mu$ is subadditive $\implies I_{\text{sum}}^\mu(A) \leq 0$ for $\forall A \subseteq N$, $\mu$ is strict subadditive $\implies I_{\text{sum}}^\mu(A) < 0$ for $\forall A \subseteq N$, $\lvert A \rvert \geq 2$.

Proof. It can be obtained from Definitions 1 and 7.

Proposition 5. Let $\mu$ be a monotone measure on $N$, if $\mu$ is subadditive then the sum interaction index of any subset of $N$ is nonpositive, and if $\mu$ is strict subadditive then the sum interaction index of the nonempty nonsingleton subset of $N$ is negative, i.e., $\mu$ is subadditive $\implies I_{\text{sum}}^\mu(A) \leq 0$ for $\forall A \subseteq N$, $\mu$ is strict subadditive $\implies I_{\text{sum}}^\mu(A) < 0$ for $\forall A \subseteq N$, $\lvert A \rvert \geq 2$.

Proof. It can be obtained from Definitions 1 and 7.

Example 3. Let $N = \{1, 2, 3\}$, $\mu_6$ is a strict subadditive monotone measure on $N$. $\mu_6$ and its Shapley interaction index and sum interaction index are given.
positive: \{1, 2, 3\} are all negative. However, the Shapely interaction values of the subsets with cardinality equal to 2 or 3 are all positive. However, the sum interaction index values of all subsets is zero.

Table 5

<table>
<thead>
<tr>
<th>A</th>
<th>(\mu_6(A))</th>
<th>(I_{\text{Sh}}^6(A))</th>
<th>(I_{\text{sum}}^6(A))</th>
<th>A</th>
<th>(\mu_6(A))</th>
<th>(I_{\text{Sh}}^6(A))</th>
<th>(I_{\text{sum}}^6(A))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\emptyset)</td>
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<td>0.514</td>
<td>0</td>
<td>({1, 2})</td>
<td>0.81</td>
<td>0.025</td>
<td>-0.235</td>
</tr>
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<td>0.4</td>
<td>0.324</td>
<td>0</td>
<td>{1, 3}</td>
<td>0.45</td>
<td>-0.035</td>
<td>0.556</td>
</tr>
<tr>
<td>{2}</td>
<td>0.6</td>
<td>0.503</td>
<td>0</td>
<td>{2, 3}</td>
<td>0.61</td>
<td>-0.075</td>
<td>0.475</td>
</tr>
<tr>
<td>{3}</td>
<td>0.3</td>
<td>0.173</td>
<td>0</td>
<td>{1, 2, 3}</td>
<td>1</td>
<td>0.430</td>
<td>-0.057</td>
</tr>
</tbody>
</table>

Table 6

<table>
<thead>
<tr>
<th>A</th>
<th>(\mu_7(A))</th>
<th>(I_{\text{Sh}}^7(A))</th>
<th>(I_{\text{sum}}^7(A))</th>
<th>A</th>
<th>(\mu_7(A))</th>
<th>(I_{\text{Sh}}^7(A))</th>
<th>(I_{\text{sum}}^7(A))</th>
</tr>
</thead>
<tbody>
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<td>(\emptyset)</td>
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<td>0.461</td>
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<td>0.435</td>
<td>0</td>
<td>{2, 3}</td>
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<td>-0.045</td>
<td>0.098</td>
</tr>
<tr>
<td>{3}</td>
<td>0.30</td>
<td>0.280</td>
<td>0</td>
<td>{1, 2, 3}</td>
<td>1</td>
<td>-0.210</td>
<td>0.157</td>
</tr>
</tbody>
</table>

in Table 5. One can see that the sum interaction index values of the subsets with cardinality equal to 2 or 3 are all negative. However, the Shapely interaction index values of the subsets \{1, 2\} and \{1, 2, 3\} are positive: \(I_{\text{Sh}}^6(\{1, 2\}) = 0.025\) and \(I_{\text{Sh}}^6(\{1, 2, 3\}) = 0.430\). The main reason of the difference between the values of the two indices is that, in essence, the sum interaction index reflects a ‘comprehensive’ interaction of the decision criteria while the Shapely interaction index reflects a simultaneous interaction of them.

**Proposition 6.** Let \(\mu\) be a monotone measure on \(N\), if \(\mu\) is superadditive then the sum interaction index of any subset of \(N\) is nonnegative, and if \(\mu\) is strict superadditive then the sum interaction index of the nonempty nonsingleton subset of \(N\) is positive, i.e., \(\mu\) is superadditive \(\Rightarrow I_{\text{sum}}^\mu(A) \geq 0\) for \(\forall A \subseteq N\), \(\mu\) is strict superadditive \(\Rightarrow I_{\text{sum}}^\mu(A) > 0\) for \(\forall A \subseteq N\) and \(|A| \geq 2\).

**Proof.** It can be obtained from Definitions 1 and 7.

**Example 4.** Let \(N = \{1, 2, 3\}\), \(\mu_7\) is a strict superadditive monotone measure on \(N\), as given in Table 6. Its Shapely interaction index and sum interaction index are also listed in Table 6. One can see that the sum interaction index values of the subsets with cardinality equal to 2 or 3 are all positive. However, the Shapely interaction index values of the subsets \{1, 3\}, \{2, 3\} and \{1, 2, 3\} are negative: \(I_{\text{Sh}}^7(\{1, 3\}) = -0.065\), \(I_{\text{Sh}}^7(\{1, 3\}) = -0.045\) and \(I_{\text{Sh}}^7(\{1, 2, 3\}) = -0.210\).

**Proposition 7.** Let \(\mu_s\) be a symmetric monotone measure on \(N\), then the sum interaction index \(I_{\text{sum}}^\mu_s\) is also symmetric, i.e., \(I_{\text{sum}}^\mu_s(A)\) depends only on \(|A|\) for any \(A \subseteq N\). That is, for \(\forall A, B \subseteq N\), if \(|A| = |B|\) then \(I_{\text{sum}}^\mu_s(A) = I_{\text{sum}}^\mu_s(B)\).

**Proof.** Since \(\mu_s(A) = \mu_s(B)\) for \(\forall A, B \subseteq N\) and \(|A| = |B|\), we have that \(\sum_{C \subseteq A} \mu(C) = \sum_{D \subseteq B} \mu(D)\) for \(\forall A, B \subseteq N\) and \(|A| = |B|\). By definition 7, we have \(I_{\text{sum}}^\mu_s(A) = I_{\text{sum}}^\mu_s(B)\) if \(|A| = |B|\). That is, \(I_{\text{sum}}^\mu_s\) is symmetric.

The converse of the proposition 7 is not true because that the index of single decision criterion is always zero no matter its monotone measure value. However, we can have the following proposition.

**Proposition 8.** Let \(\mu\) be a monotone measure on \(N\), if \(\mu(\{i\}) = \mu(\{j\})\) for \(\forall i, j \in N\), and the sum interaction index \(I_{\text{sum}}^\mu\) is symmetric, then the monotone measure \(\mu\) is also symmetric.

**Proof.** From definition 7, we have \(\mu(A) = \mu(B)(2^{|A|} - 1)\) if \(I_{\text{sum}}^\mu(A) = 0\) and \(|A| \geq 2\). If \(I_{\text{sum}}^\mu(A) \neq 0\), we have

\[
I_{\text{sum}}^\mu(A) = \frac{2^{|A|} - 1}{2^{|A|} - 1} \mu(A) - \mu(A)
\]

\[
= \frac{2^{|A|} - 1}{2^{|A|} - 1} \mu(A) - \mu(A)
\]

\[
= \frac{\sum_{C \subseteq A} \mu(C)}{2^{|A|} - 1} \mu(A) - \mu(A)
\]

\[
= 1 - \frac{\sum_{C \subseteq A} \mu(C)}{2^{|A|} - 1} \mu(A) - \mu(A)
\]

\[
= 1 - \frac{\sum_{C \subseteq A} \mu(C)}{2^{|A|} - 1} \mu(A) - \mu(A)
\]

Since \(\mu(\{i\}) = \mu(\{j\})\) for \(\forall i, j \in N\), we have that \(\mu(A) = \mu(B)\) for \(|A| = |B| = 1\). Then by the mathematical induction, if \(I_{\text{sum}}^\mu\) is symmetric, we can get \(\mu\) is symmetric.

**4. Conclusions**

In this paper, we propose the sum interaction index from the perspective of multicriteria decision analysis. The sum interaction index has a relative simpler structure and all of the index values are normalized into a common interval \([-1, 1]\), which means that it is feasible and easy to compare the degrees of the interactions of different subsets, even of the subsets with different cardinalities. The sum interaction index also satisfies some intuitive properties, e.g., the sum interaction index values of all subsets is zero.
if and only if the corresponding monotone measure is additive; the sum interaction index values of all subsets with respect to a subadditive (resp. superadditive) monotone measure are always nonpositive (resp. nonnegative). The illustrative examples given in this paper demonstrate that the sum interaction index is a reasonable and feasible tool to measure the interaction phenomenon of the multiple decision criteria. In the future, we will discuss the properties of the sum interaction index associated with some particular families of monotone measures.

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