The sum interaction indices of some particular families of monotone measures

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Abstract. In the literature [40], the authors proposed the sum interaction index and discussed its basic mathematical properties. And by some comparison analyses, the authors demonstrated that the sum interaction index can be taken as an alternative of the probabilistic interaction index to measure the interaction phenomenon of the multiple decision criteria. In this paper, we further investigate the properties of the sum interaction index with respect to some particular families of monotone measures, such as the $\lambda$-monotone measures, the possibility monotone measures, the $k$-additive monotone measures, the $p$-symmetric monotone measures, and the $k$-tolerant and $k$-intolerant monotone measures. Some illustrative examples are also given to show the comparison analyses between the sum interaction index and the probabilistic interaction index.

Keywords: Multicriteria decision analysis, monotone measure, shapley interaction index, sum interaction index, $k$-additive monotone measure

1. Introduction

In the framework of the monotone measure based multicriteria decision analysis, the importance of a decision criteria subset is directly described by its measure value and the interaction among decision criteria is usually represented by the probabilistic simultaneous interaction index, such as the Shapley interaction index [14], the Banzhaf interaction index [21], the chaining interaction index [21, 31] and the Möbius representation or called the internal interaction index [8, 16]. Although the probabilistic simultaneous interaction indices have some good mathematical properties and distinctive axiomatic characteristics (see, e.g., [12, 21, 24]), they sometimes fail to satisfy some common intuitive properties in the context of multicriteria decision analysis. For example, a strict superadditive monotone measure basically represents a certain positive interaction case of all decision criteria, but any type of simultaneous interaction index of a strict superadditive monotone measure is not guaranteed to be positive. In view of this lack of the probabilistic interaction index, in the literature [40], we proposed monotone measure sum based interaction index, or called the sum interaction index for short, as a feasible alternative of the probabilistic interaction index to describe the interaction phenomenon of decision criteria. The sum interaction index of a subset is basically determined by the sum of measure values of all subsets of the subset. The sum interaction index has a relatively simpler structure and all of the index values are normalized into a common interval $[-1, 1]$, which means that it is feasible and easy to compare the degrees of the interactions of different subsets, even of the subsets with different cardinalities. The sum interaction index also satisfies some intuitive properties, e.g., the sum interaction index values of all subsets is zero if and only if the corresponding monotone measure is additive; the sum interaction index values of all subsets with respect to a subadditive (resp. superadditive, strict subadditive, and strict superadditive) monotone measure...
measure are always nonpositive (resp. nonnegative, negative, and positive).

In the monotone measure theory, some particular families of monotone measures are introduced to reduce the exponential complexity inherent in the construction process of monotone measure [17, 18, 42]. The most famous ones can include the λ- monotone measures [36, 37], the possibility monotone measures [22, 23, 44], the k-additive monotone measures [16], the p-symmetric monotone measures [32], and the k-tolerant and k-intolerant monotone measures [29, 30]. These above particular families usually have the relatively simpler structure and hence can effectively reduce the number of coefficients in the identification process of the monotone measure on criteria set, meanwhile the ability to flexibly describe the interactions among decision criteria is well preserved. Hence, these particular families have been widely accepted in the field of multicriteria decision analysis (see, e.g., [1, 2, 4, 6, 15, 17, 25, 33, 42]). In view of this point, it should be useful for the theory and application of the monotone measure based multicriteria decision analysis to discuss the properties and characteristics of the sum interaction indices with respect to the particular families of monotone measures.

This paper is organized as follows. After the introduction, we present some knowledge about the monotone measure, the probabilistic interaction indices and the sum interaction indices in Section 2. In Section 3, we show some properties of the sum interaction indices with respect to the particular families of monotone measures, and several comparison analyses with the probabilistic interaction indices are also presented. Finally, we conclude the paper and discuss future work in Section 4.

For convenience, let \( N = \{1, 2, \ldots, n\}, n \geq 2 \), be the decision criteria set, \( P(N) \) be the power set of \( N \).

2. Preliminaries

**Definition 1.** (See [10, 11, 17, 34, 36, 37, 39, 42]) A monotone measure (or called fuzzy measure, non-additive measure, capacity) on \( N \) is a set function \( \mu : P(N) \rightarrow [0, 1] \) such that

(i) \( \mu(\emptyset) = 0, \mu(N) = 1; \)

(ii) for \( \forall A, B \subseteq N, A \subseteq B \) implies \( \mu(A) \leq \mu(B) \).

Furthermore, a monotone measure \( \mu \) on \( N \) is said to be additive (resp. superadditive, strict superadditive, subadditive, and strict subadditive), if \( \mu(A \cup B) = (\mu(A) \leq \mu(B) \) for \( \forall A, B \subseteq N \) and \( A \cap B = \emptyset; \) to be symmetric or cardinality based if \( \mu(A) \) depends only on \( |A| \) for any \( A \subseteq N \), where \( |A| \) is the cardinality of set \( A \).

One can see that a monotone measure is a normal monotone set function which vanishes at the empty set [22]. In the framework of the multicriteria decision analysis, the number \( \mu(A) \) can be interpreted as the importance of the subset \( A \subseteq N \), and the monotonicity with respect to set inclusion (condition (ii) in Definition 1) means that the importance of a subset of criteria cannot decrease when new criteria are added to it [17]. This monotonicity constraint enables the monotone measure to flexibly represent the various kinds of interactions among the decision criteria, ranging from substitutivity (negative interaction) to complementarity (positive interaction) [13, 22]. Generally speaking, the additive monotone measure means that the decision criteria are all independent to each other, i.e., the interaction of arbitrary group of decision criteria should preferably be zero. The strict superadditive (resp. strict subadditive, superadditive, and subadditive) monotone measure means that, to a certain extent, all the decision criteria can be considered as mutually complementary (substitutive, nonsubstitutive, and noncomplementary), so a certain kind of positive (resp. negative, nonnegative, and nonpositive) interaction exists among arbitrary group of decision criteria.

When using a monotone measure to model the importance of the subsets of decision criteria, a suitable aggregation function is the Choquet integral [5, 17].

**Definition 2.** (See [10, 13, 14, 37, 39, 43]) Let \( x \) be a real-valued function on \( N \), \( x := (x_{1}, x_{2}, \ldots, x_{n}) \), the Choquet integral of \( x \) with respect to a monotone measure \( \mu \) on \( N \) is defined as

\[
C_{\mu}(x) = \sum_{i=1}^{n} [x_{(i)} - x_{(i-1)}] \mu(N_{(i)})
\]

where the parentheses used for indices represent a permutation on \( N \) such that \( x_{(1)} \leq \cdots \leq x_{(n)} \), \( x_{(0)} = 0, N_{(i)} = \{i, \ldots, n\} \).

The Choquet integral has some good aggregation properties, such as idempotence, compensativeness and comonotonic additivity [28, 39]. The Choquet integral has been widely used as an aggregation tool in decision making [7, 9, 13, 14, 18, 27, 33].
In the context of the monotone measure and Choquet integral based multicriteria decision analysis, the overall importance of a decision criterion is measured by the probabilistic value and the simultaneous interaction among multiple decision criteria is usually measured by the probabilistic interaction index.

Definition 3. (See [12, 26, 38]) Let \( \mu \) be a monotone measure on \( N \), the probabilistic value of a criterion \( i \in N \) with respect to \( \mu \) is defined as

\[
I_\mu^p(i) = \sum_{B \subseteq N \setminus \{i\}} p_B(N)[\mu(B \cup \{i\}) - \mu(B)],
\]

where, for any \( i \in N \), the family of coefficients \( \{ p_B(N) \}_{B \subseteq N \setminus \{i\}} \) forms a probability distribution on \( P(N \setminus \{i\}) \). Additionally, for any \( i \in N \) and \( B \subseteq N \setminus \{i\} \), if the coefficient \( p_B(N) \) depends only on \( |\{i\}|, |B| \) and \( |N| \), i.e., the cardinalities of the subsets \{i\}, B and N, then the probabilistic value is called as a cardinal probabilistic value or a semivalue.

The two most famous probabilistic values, the Shapley value [35] and the Banzhaf value [3], are both the cardinal probabilistic values. Let \( \mu \) be a monotone measure on \( N \), the Shapley value of a criterion \( i \in N \) with respect to \( \mu \) is defined as

\[
I_\mu^{Sh}(i) = \sum_{B \subseteq N \setminus \{i\}} |B|!(|N| - |B| - 1)! |N|! \left[ \mu(B \cup \{i\}) - \mu(B) \right],
\]

the Banzhaf value of a criterion \( i \in N \) with respect to \( \mu \) is defined as

\[
I_\mu^{Ba}(i) = \sum_{B \subseteq N \setminus \{i\}} \frac{1}{2(|N| - |A|)} \left[ \mu(B \cup \{i\}) - \mu(B) \right].
\]

Definition 4. (See [12, 21]) Let \( \mu \) be a monotone measure on \( N \), the probabilistic interaction index of any subset \( A \subseteq N \) with respect to \( \mu \) is defined as

\[
I_\mu^p(A) = \sum_{B \subseteq N \setminus A} p_B^A(N) \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right),
\]

where, for any \( A \subseteq N \), the family of coefficients \( \{ p_B^A(N) \}_{B \subseteq N \setminus A} \) forms a probability distribution on \( P(N \setminus A) \). Additionally, for any \( A \subseteq N \) and \( B \subseteq N \setminus A \), if the coefficient \( p_B^A(N) \) depends only on \(|A|, |B| \) and \(|N| \), the probabilistic interaction index is called as the cardinal probabilistic interaction index.

By comparing Definitions 3 and 4, one can see that the probabilistic interaction index is a natural generalization of the probabilistic value. The following definition gives four well-known probabilistic interaction indices.

Definition 5. (See [8, 12, 16, 19, 20, 31]) Let \( \mu \) be a monotone measure on \( N \), the Shapley interaction index of any subset \( A \subseteq N \) with respect to \( \mu \) is defined as

\[
I_\mu^{Sh}(A) = \sum_{B \subseteq N \setminus A} \frac{1}{(|N| - |A| + 1)} \left( \frac{|N| - |A|}{|B|} \right)^{-1} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right),
\]

the Banzhaf interaction index of any subset \( A \subseteq N \) with respect to \( \mu \) is defined as

\[
I_\mu^{Ba}(A) = \sum_{B \subseteq N \setminus A} \frac{1}{2(|N| - |A|)} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right),
\]

the chaining interaction index of any subset \( A \subseteq N \), \( A \neq \emptyset \), with respect to \( \mu \) is defined as

\[
I_\mu^{ch}(A) = \sum_{B \subseteq N \setminus A} \frac{|A|}{|A| + |B|} \left( \frac{|N|}{|A| + |B|} \right)^{-1} \left( \sum_{C \subseteq A} (-1)^{|A\setminus C|} \mu(C \cup B) \right),
\]

the Möbius representation of any subset \( A \subseteq N \) with respect to \( \mu \) is defined as

\[
m_\mu(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B).
\]

The Shapley interaction index and the chaining interaction index both extend the Shapley value [12, 21, 35] while the Banzhaf interaction index just extends the Banzhaf value [3]. It should be pointed out that, among the four cardinal probabilistic interaction indices, the Shapley interaction index has some distinctive axiomatic properties, such as linearity axiom, dummy axiom, symmetry axiom, efficiency axiom and recursive axiom [12], and hence the Shapley interaction index is the most suitable one to measure the simultaneous interaction among the multicriteria decision criteria [12, 20].

However, the simultaneous interaction indices sometimes fail to satisfy some intuitive properties in the context of multicriteria decision analysis. For example, the superadditive (resp. subadditive) relationship in the monotone measure theory basically describes a certain kind of nonnegative (resp.
nonpositive) interaction among the multiple decision criteria, so it is a natural requirement that the corresponding interaction index values of all the criteria subsets can be nonnegative (resp. nonpositive). Unfortunately, any type of the probabilistic simultaneous interaction index fails to satisfy these intuitive properties when the universal set consists of three or more decision criteria. Another main imperfection is that the simultaneous interaction index values of the subsets that with different cardinalities usually have the different ranges. For example, the ranges of the Shapley simultaneous interaction index of two criteria, three criteria, four criteria and five criteria are \([-1, 1], [-2, 1], [-3, 3] \text{ and } [-4, 6]\), respectively [40]. So, it becomes difficult to directly estimate the interaction degree from the value of the simultaneous interaction index, and it is rather confused to directly compare the values of two simultaneous interaction indices with different orders.

In view of these imperfections of the probabilistic interaction indices, including the Shapley simultaneous interaction index, in the literature [40], we proposed the notion of the sum interaction index and discussed its basic mathematical properties.

**Definition 6.** [40] Let \( \mu \) be a monotone measure on \( N \), for \( \forall A \subseteq N \), the sum interaction index of \( A \) is defined as

\[
I_{\text{sum}}(A) = \frac{2^{|A|} - 1}{2^{|A|} - 1} \mu(A) - \sum_{B \subseteq A} \mu(B)
\]

if \( 2^{|A|} - 1 \mu(A) - \mu(A) \neq 0 \), and \( I_{\text{sum}}(A) = 0 \) otherwise, where \( \sum_{B \subseteq A} \mu(B) \) is the cardinality of the subset \( A \).

One can see that \( I_{\text{sum}}(A) = 0 \) if and only if

\[
2^{|A|} - 1 \mu(A) - \sum_{B \subseteq A} \mu(B) = 0.
\]

**Proposition 1.** [40] The sum interaction index of any singleton set as well as of the empty set is equal to 0, i.e., \( I_{\text{sum}}(\{i\}) = 0 \) for \( \forall i \in N \), and \( I_{\text{sum}}(\emptyset) = 0 \).

**Proposition 2.** [40] Let \( \mu \) be a monotone measure on \( N \), then \( \mu \) is additive \( \Leftrightarrow I_{\text{sum}}(A) = 0 \) for \( \forall A \subseteq N \).

The Shapley interaction index also has a similar property, i.e., the dummy axiom [12, 20, 21]. Let \( \mu \) be a monotone measure on \( N \), a decision criterion \( i \in N \) is said to be dummy [21] if \( \mu(A \cup \{i\}) = \mu(A) + \mu(\{i\}) \) for any \( A \subseteq N \backslash \{i\} \). The dummy axiom is that if \( i \in N \) is a dummy, then \( I_{\text{sh}}^{\mu}(\{i\}) = \mu(\{i\}) \), \( I_{\text{sh}}^{\mu}(A \cup \{i\}) = 0 \) for any \( A \subseteq N \backslash \{i\} \) such that \( A \neq \emptyset \). The dummy axiom means that the overall importance of the dummy criterion is equal to its monotone measure value and that the dummy criterion does not interact with any other subset [12, 20, 21]. That is, the Shapley interaction index, or the simultaneous interaction, of the subset that contains a dummy criterion will equal to zero. Since for the additive monotone measure, every decision criterion can be regarded as a dummy criterion, then we have that \( \mu \) is additive \( \Rightarrow I_{\text{sh}}^{\mu}(A) = 0 \) for \( \forall A \subseteq N \), \( |A| \geq 2 \).

It should be mentioned that converse of this statement is not true. Furthermore, the sum interaction index of the subset that containing the dummy criteria is not always zero.

**Proposition 3.** [40] Let \( \mu \) be a monotone measure on \( N \), \( \mu \) is subadditive \( \Rightarrow I_{\text{sum}}^{\mu}(A) \leq 0 \) for \( \forall A \subseteq N \), \( \mu \) is strict subadditive \( \Rightarrow I_{\text{sum}}^{\mu}(A) < 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \).

**Proposition 4.** [40] Let \( \mu \) be a monotone measure on \( N \), \( \mu \) is superadditive \( \Rightarrow I_{\text{sum}}^{\mu}(A) \geq 0 \) for \( \forall A \subseteq N \), \( \mu \) is strict superadditive \( \Rightarrow I_{\text{sum}}^{\mu}(A) > 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \).

**Proposition 5.** [40] Let \( \mu_\lambda \) be a symmetric monotone measure on \( N \), for \( \forall A, B \subseteq N \), if \( |A| = |B| \) then \( I_{\text{sum}}^{\mu}(A) = I_{\text{sum}}^{\mu_\lambda}(B) \).

The converse of Proposition 5 is not true because that the index of single decision criterion is always zero no matter its monotone measure value. However, we can have the following proposition.

**Proposition 6.** [40] Let \( \mu \) be a monotone measure on \( N \), if \( \mu(\{i\}) = \mu(\{j\}) \) for \( \forall i, j \in N \), and the sum interaction index \( I_{\text{sum}}^{\mu} \) is symmetric, then the monotone measure \( \mu \) is also symmetric.

### 3. The sum interaction indices of the particular monotone measures

In this section, we discuss the properties of the sum interaction index associated with some particular families of monotone measures.

From Definition 1, we know that any subset’s monotone measure value should be defined in order to obtain a monotone measure on the decision criteria set, which means that there exists an inherent exponential complexity [13] in its construction process. In order to reduce this exponential complexity, some particular families of monotone measures have been proposed [17, 18, 42], such as the \( \lambda \)-measures [36, 37], the possibility measures [22, 23, 44], the
3.1. The \( \lambda \)-measure

**Definition 7.** (See [36, 37]) Let \( \mu \) be a monotone measure on \( N \), \( \mu \) is called a \( \lambda \)-measure if \( \mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A)\mu(B) \) for all disjoint subsets \( A, B \subseteq N \), where \( \lambda \in (-1, +\infty) \).

To identify a \( \lambda \)-measure, denoted as \( \mu_{\lambda} \), on \( N \), we only need to define the measure value of any decision criterion, \( \mu_{\lambda}[i], i \in N \) [36, 37]. Furthermore, by the following equation, we can get the coefficient \( \lambda \) [36, 37]: \( \prod_{i=1}^{N} (1 + \lambda \mu_{\lambda}[i]) = 1 + \lambda \).

Let \( \mu_{\lambda} \) be a \( \lambda \)-measure on \( N \), then for \( \forall A \subseteq N \),

\[
2^{|A|-1} \mu_{\lambda}(A) - \sum_{B \subseteq A} \mu_{\lambda}(B)
\]

\[
= \frac{1}{2} \left( 2^{|A|} \mu_{\lambda}(A) - \sum_{B \subseteq A} \mu_{\lambda}(B) \right)
\]

\[
= \frac{1}{2} \sum_{B \subseteq A} (\mu_{\lambda}(A) - 2\mu_{\lambda}(B))
\]

\[
= \frac{1}{2} \sum_{B \subseteq A} (\mu_{\lambda}(A) - \mu_{\lambda}(B) - \mu_{\lambda}(A \setminus B))
\]

\[
= \sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B).
\]

From Definition 6, we have

\[
I_{\text{sum}}^{\mu_{\lambda}}(A) = \frac{\sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B)}{2^{|A|-1} \mu_{\lambda}(A) - \mu_{\lambda}(A)}
\]

if \( \sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B) \neq 0 \), and \( I_{\text{sum}}^{\mu_{\lambda}}(A) = 0 \) if

\[
\sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B) = 0.
\]

Hence, we can directly have the following proposition.

**Proposition 7.** Let \( \mu_{\lambda} \) be a \( \lambda \)-measure on \( N \), then \( \lambda = 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) = 0 \) for \( \forall A \subseteq N \), \( \lambda > 0 \Rightarrow I_{\text{sum}}^{\mu_{\lambda}}(A) \geq 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \), \( \lambda < 0 \Rightarrow I_{\text{sum}}^{\mu_{\lambda}}(A) \leq 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \).

Furthermore, if \( \mu_{\lambda}(\{i\}) \neq 0 \) for \( \forall i \in N \), then \( \lambda > 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) > 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \), \( \lambda < 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) < 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \).

**Proof.** For the \( \lambda \)-measure \( \mu_{\lambda} \) on \( N \), by Definition 7, if \( \lambda = 0 \), then \( \mu_{\lambda} \) is just an additive measure, then by Proposition 2, we can have \( \lambda = 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) = 0 \) for \( \forall A \subseteq N \). If \( \lambda > 0 \), since \( \mu_{\lambda}(A) \geq 0 \) for \( \forall A \subseteq N \), we can have \( \sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B) \geq 0 \), then by the expression of the sum interaction with respect to a \( \lambda \)-measure, we get \( I_{\text{sum}}^{\mu_{\lambda}}(A) \geq 0 \). Similarly, we can get that \( \lambda < 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) \leq 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \). Furthermore, if \( \mu_{\lambda}(\{i\}) \neq 0 \) for \( \forall i \in N \), then by Definition 7, we have \( \mu_{\lambda}(A) \neq 0 \) for \( \forall A \neq \emptyset \subseteq N \), which means that \( \sum_{B \subseteq A} \frac{\lambda}{2} \mu_{\lambda}(B)\mu_{\lambda}(A \setminus B) > \) (resp. \( < \)) 0 for \( \forall A \subseteq N \) and \( |A| \geq 2 \) if and only if \( \lambda > \) (resp. \( < \)) 0. That is, we can have that if \( \mu_{\lambda}(\{i\}) \neq 0 \) for \( \forall i \in N \), then \( \lambda > 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) > 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \), \( \lambda < 0 \iff I_{\text{sum}}^{\mu_{\lambda}}(A) < 0 \) for \( \forall A \subseteq N \) and \( |A| \geq 2 \).

**Example 1.** Let \( N = \{1, 2, 3, 4\} \), \( \mu_{-0.9} \) is a \( \lambda \)-measure on \( N \) with \( \lambda = -0.9 \), as given in Table 1. Its Shapley interaction index and sum interaction index are also listed in Table 1. We can take the subset \{1, 2, 3\} as an example to show the calculation process of the \( \lambda \)-measure’s sum interaction index.

<table>
<thead>
<tr>
<th>( A )</th>
<th>( \mu_{-0.9}(A) )</th>
<th>( I_{\text{Shapley}}^{\mu_{-0.9}}(A) )</th>
<th>( m_{\mu_{-0.9}}(A) )</th>
<th>( I_{\text{sum}}^{\mu_{-0.9}}(A) )</th>
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\[ \sum_{B \subseteq \{1,2,3\}} -0.9 \frac{\mu_{-0.9}(B)\mu_{-0.9}(A \setminus B)}{2} \]

\[ = -0.9\mu_{-0.9}(\{1\})\mu_{-0.9}(\{2,3\}) \]

\[ -0.9\mu_{-0.9}(\{2\})\mu_{-0.9}(\{1,3\}) \]

\[ -0.9\mu_{-0.9}(\{3\})\mu_{-0.9}(\{1,2\}) \]

\[ = -0.6754. \]

then we get \( I_{\text{sum}}^{\mu_{-0.9}}(\{1,2,3\}) = -0.2671. \)

Furthermore, from Table 1, one can see that the sum interaction index values of the subsets that with cardinality larger than or equal to 2 are all negative, since \( \lambda = -0.9 < 0 \) and \( \mu_{-0.9} \) is a strict subadditive measure. In contrast, the Shapely interaction index and Möbius representation values of some subsets are positive, see the interaction indices values of the subsets with three decision criteria.

3.2. The possibility measure

Definition 8. (See [22, 23, 44]) Let \( \mu \) be a monotone measure on \( N, \mu \) is called a possibility measure if \( \mu(A \cup B) = \mu(A) \lor \mu(B) \) for all disjoint subsets \( A, B \subseteq N. \)

To identify a possibility measure \( \mu \) on \( N, \) we also only need to define the measure value of any decision criterion, \( \mu\{i\}, i \in N \)

Let \( \mu \) be a possibility measure on \( N, \) then for \( \forall A \subseteq N, \)

\[ 2^{|A|-1}(A) - \sum_{B \subseteq A} \mu(B) = \frac{1}{2} \left( \sum_{B \subseteq A} \mu(A) - 2\mu(B) \right) \]

\[ = \frac{1}{2} \left( \sum_{B \subseteq A} \mu(A) - \mu(B) - \mu(A \setminus B) \right) = \frac{2^{|A|-1}}{2} \sum_{k=1}^{2^{|A|-1}} \mu(k), \]

where \( \mu(k) \) is the \( k \)th order statistic or \( k \)th smallest value of the family of measure values \( \{\mu(B) | B \subseteq A\}, k = 1, 2, \ldots, 2^{|A|}, \mu(\emptyset) = 0, \mu(\emptyset) = \mu(A). \)

Hence, from Definition 6, we have \( I_{\text{sum}}^{\mu}(A) = \frac{2^{|A|-1}}{2} \sum_{k=1}^{2^{|A|-1}} \mu(k) \) if \( \sum_{k=1}^{2^{|A|-1}} \mu(k) \neq 0, \)

and \( I_{\text{sum}}^{\mu}(A) = 0 \) if \( \sum_{k=1}^{2^{|A|-1}} \mu(k) = 0. \)

It should be pointed out that, from Definition 8, we can get that the possibility measure is a subadditive measure, and from the above equations we know that the sum interaction index of all subsets are nonpositive. That is consistent with Proposition 3.

Example 2. Let \( N = \{1, 2, 3, 4\}, \mu_p \) is a possibility measure on \( N, \) as given in Table 2. Its Shapely interaction index, Möbius representation, and sum interaction index are also listed in Table 2. For example, for the subset \( \{1, 2, 3\}, \)

\[ \{\mu_p(B) | B \subseteq \{1, 2, 3\} \}

\[ = \{\mu_p(\emptyset), \mu_p(\{1\}), \mu_p(\{2\}), \mu_p(\{3\}), \mu_p(\{1, 2\}), \mu_p(\{1, 3\}), \mu_p(\{2, 3\}), \mu_p(\{1, 2, 3\}) \}

\[ = \{0, 0.3, 0.4, 0.5, 0.4, 0.5, 0.5\}, \]

we can have \( \mu_p(\emptyset) = 0, \mu_p(\{1\}) = 0.3, \mu_p(\{2\}) = 0.4, \mu_p(\{3\}) = 0.5, \mu_p(\{1, 2\}) = 0.4, \) then \( I_{\text{sum}}^{\mu_p}(\{1, 2, 3\}) = -1.1 / 1.5 = 0.7333. \)

Furthermore, from Table 2, one can see that the sum interaction index values of the subsets that with cardinality equal to or larger than 2 are all negative, since the possibility measure \( \mu_p \) is a subadditive measure. In contrast, the Shapely interaction index values of some subsets are positive, see the Shapely interaction index values of the subsets with three decision criteria.

3.3. The \( k \)-additive measure

Definition 9. (See [16–18]) Let \( k \in \{1, 2, \ldots, n\}, \) a monotone measure \( \mu \) on \( N \) is said to be \( k \)-additive if its Möbius representation satisfies \( m_{\mu}(A) = 0 \) for all \( A \subseteq N \) such that \( |A| > k \) and there exists at least one subset \( A \) of \( k \) elements such that \( m_{\mu}(A) \neq 0. \)

1-additive measure is just the additive measure [16–18]. To identify a \( k \)-additive measure \( \mu \) on \( N, \) we only need to define the Möbius representation values of the subsets whose cardinalities are not larger
than $k$. That is, we only need to define $\sum_{i=1}^{k} \binom{n}{i}$

coefficients [18].

An important property of the $k$-additive measure $\mu$ on $N$ is that $I_{\hat{\mu}}(A) = 0$ for all $A \subseteq N$ such that $|A| > k$. That is, the $k$-additive measure takes account of the simultaneous interactions among at most $k$ decision criteria and ignores the higher order simultaneous interactions [16–18].

From Definition 5, we can have that [16]

$$\mu(A) = \sum_{B \subseteq A} m_{\mu}(B) \text{ for } \forall A \subseteq N.$$ 

Then we have

$$2^{|A|-1} \mu(A) - \sum_{B \subseteq A} \mu(B)$$

$$= \sum_{B \subseteq A} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu}(B)$$

From Definition 9, we know that $m_{\mu}(A) = 0$ for all $A \subseteq N$ such that $|A| > k$. So, if the monotone measure is $k$-additive, we can have

$$2^{|A|-1} \mu(A) - \sum_{B \subseteq A} \mu(B)$$

$$= \sum_{B \subseteq A, |B| \leq k} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu}(B).$$

Hence, the sum interaction index of a $k$-additive measure $\mu$ can be given as follows:

$$I_{\hat{\mu}}^\mu(A) = \frac{\sum_{B \subseteq A, |B| \leq k} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu}(B)}{(2^{|A|-1} - 1) \sum_{B \subseteq A, |B| \leq k} m_{\mu}(B)}$$

if $\sum_{B \subseteq A, |B| \leq k} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu}(B) \neq 0$, and $I_{\hat{\mu}}^\mu(A) = 0$ if $\sum_{B \subseteq A, |B| \leq k} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu}(B) = 0$.

One can see that $I_{\hat{\mu}}^\mu(A)$ of a $k$-additive measure $\mu$ is usually nonzero when $|A| > k$.

Example 3. (Adapted from [1]) Let $N = \{1, 2, 3, 4\}$, $\mu_2$ is a 2-additive measure on $N$, as given in Table 3. Its Möbius representation, Shapley interaction index and sum interaction index are also listed in Table 3.

For example, for the subset $\{1, 2, 3\}$, since

$$\sum_{B \subseteq A, |B| \leq 2} (2^{|A|-1} - 2^{|A \setminus B|}) m_{\mu_2}(B)$$

$$= 2m_{\mu_2}(\{1, 2\}) + 2m_{\mu_2}(\{1, 3\}) + 2m_{\mu_2}(\{2, 3\}) = 0.272,$$

we have $I_{\hat{\mu}}^\mu_2(\{1, 2, 3\}) = 0.272/2.643 = 0.1029$.

From Table 3, one can see the Shapley interaction index and Möbius representation values of the subsets with cardinality larger than 2 are all zero since $\mu_2$ is a 2-additive measure. In contrast, the sum interaction index values of the subsets with cardinality larger than 2 are all nonzero. From the perspective of multicriteria decision analysis, it is hard to explain the case that the interactions among three or more decision criteria are all zero regardless of the kinds and the extents of interactions of all pairs of decision criteria. In this sense, the sum interaction index is more suitable to represent the interaction phenomenon associated with the 2-additive measure $\mu_2$.

3.4. The $p$-symmetric measure

**Definition 10.** (See [18, 32]) Let $\mu$ be a monotone measure on $N$, a subset $A \subseteq N$ is a subset of indifference with respect to $\mu$ if for $\forall B_1, B_2 \subseteq A$, $|B_1| = |B_2|$, we have $\mu(C \cup B_1) = \mu(C \cup B_2)$ for $\forall C \subseteq N \setminus A$.

Any subset of a subset of indifference is also a subset of indifference, and any singleton subset is a subset of indifference [18, 32].

**Definition 11.** (See [18, 32]) A monotone measure $\mu$ on $N$ is said to be $p$-symmetric if the coarsest partition of $N$ into subsets of indifference contains exactly $p$ subsets $A_1, A_2, \ldots, A_p$ where $A_i$ is a subset of indifference, $A_i \cap A_j = \emptyset$, $\bigcup_{i=1}^{p} A_i = N$, $i, j = 1, 2, \ldots, p$, and a partition $\pi$ is coarser than another partition $\pi'$ if all subsets of $\pi$ are union of some subsets of $\pi'$. The partition $\{A_1, A_2, \ldots, A_p\}$ is called the basis of $\mu$. 

### Table 3

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\mu_2(A)$</th>
<th>$I_{\hat{\mu}}^{\mu_2}(A)$</th>
<th>$m_{\mu_2}(A)$</th>
<th>$I_{\hat{\mu}}^{\mu}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
<td>0.4835</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>${1}$</td>
<td>0.377</td>
<td>0.4375</td>
<td>0.3770</td>
<td>0</td>
</tr>
<tr>
<td>${2}$</td>
<td>0.221</td>
<td>0.3295</td>
<td>0.2210</td>
<td>0</td>
</tr>
<tr>
<td>${3}$</td>
<td>0.147</td>
<td>0.0955</td>
<td>0.1470</td>
<td>0</td>
</tr>
<tr>
<td>${4}$</td>
<td>0.156</td>
<td>0.1375</td>
<td>0.1560</td>
<td>0</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>0.856</td>
<td>0.2580</td>
<td>0.2580</td>
<td>0.3014</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>0.544</td>
<td>0.0200</td>
<td>0.0200</td>
<td>0.0368</td>
</tr>
<tr>
<td>${1, 4}$</td>
<td>0.376</td>
<td>-0.1570</td>
<td>-0.1570</td>
<td>-0.4176</td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>0.226</td>
<td>-0.1420</td>
<td>-0.1420</td>
<td>-0.6283</td>
</tr>
<tr>
<td>${2, 4}$</td>
<td>0.478</td>
<td>0.1010</td>
<td>0.1010</td>
<td>0.2113</td>
</tr>
<tr>
<td>${3, 4}$</td>
<td>0.322</td>
<td>0.0190</td>
<td>0.0190</td>
<td>0.0590</td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>0.881</td>
<td>0.0000</td>
<td>0</td>
<td>0.1029</td>
</tr>
<tr>
<td>${1, 2, 4}$</td>
<td>0.956</td>
<td>0.0000</td>
<td>0</td>
<td>0.1409</td>
</tr>
<tr>
<td>${1, 3, 4}$</td>
<td>0.562</td>
<td>0.0000</td>
<td>0</td>
<td>-0.1400</td>
</tr>
<tr>
<td>${2, 3, 4}$</td>
<td>0.502</td>
<td>0.0000</td>
<td>0</td>
<td>-0.0292</td>
</tr>
<tr>
<td>${1, 2, 3, 4}$</td>
<td>1.000</td>
<td>0.0000</td>
<td>0</td>
<td>0.0566</td>
</tr>
</tbody>
</table>
1-symmetric measure is just the symmetric measure [18, 32]. Let \( \{A_1, A_2, \ldots, A_p\} \) be the basis of a \( p \)-symmetric measure \( \mu \) on \( N \), then any subset \( B \subseteq N \) can be identified with a \( p \)-dimensional vector \((b_1, b_2, \ldots, b_p)\), where \( b_i = |B \cap A_i|, i = 1, 2, \ldots, p \) [18, 32]. That is, a \( p \)-symmetric measure needs \( \prod_{i=1}^{p} (|A_i| + 1) \) coefficients to be defined [18, 32].

For a \( p \)-symmetric measure \( \mu \) on \( N \), we can get the following property of the sum interaction index.

**Proposition 8.** Let \( \mu \) be a \( p \)-symmetric measure on \( N \) with the basis \( \{A_1, A_2, \ldots, A_p\} \), then the sum interaction index of \( \forall B \subseteq N, I_{\text{sum}}^p(B) \), as well as the cardinal probabilistic interaction index of \( \forall B \subseteq N, I_{\text{lp}}^p(B) \), depends only on the \( p \)-dimensional vector \((b_1, b_2, \ldots, b_p)\), where \( b_i = |B \cap A_i|, i = 1, 2, \ldots, p \).

**Proof.** Since \( \mu \) is a \( p \)-symmetric measure on \( N \) with the basis \( \{A_1, A_2, \ldots, A_p\} \), then for \( \forall B \subseteq N \), the value \( \mu(B) \) is definitely identified by the \( p \)-dimensional vector \((b_1, b_2, \ldots, b_p)\), where \( b_i = |B \cap A_i|, i = 1, 2, \ldots, p \). So, we can write \( B = (b_1, b_2, \ldots, b_p) \) and \( \mu(B) = \mu(b_1, b_2, \ldots, b_p) \). By Definition 6, we have

\[
I_{\text{sum}}^p(B) = \frac{2^{|B|-1} \mu(B) - \sum_{C \subseteq B} \mu(C)}{2^{|B|-1} \mu(B) - \mu(B)}
\]

\[
= \frac{\sum_{i=1}^{p} b_i \mu(b_1, b_2, \ldots, b_p) - \sum_{i=1}^{p} \sum_{c=0}^{b_i} \mu(c_1, c_2, \ldots, c_p)}{\sum_{i=1}^{p} b_i \mu(b_1, b_2, \ldots, b_p) - \mu(b_1, b_2, \ldots, b_p)}
\]

if \( \sum_{i=1}^{p} b_i \mu(b_1, b_2, \ldots, b_p) - \mu(b_1, b_2, \ldots, b_p) \neq 0 \) and \( I_{\text{sum}}^p(B) = 0 \) otherwise. That is, with the same \( p \)-dimensional vector \((b_1, b_2, \ldots, b_p)\), we can have the same interaction index value. Similarly, for the cardinal probabilistic interaction index of \( \forall B \subseteq N \),

\[
I_{\text{lp}}^p(B) = \sum_{C \subseteq N \setminus B} p_C^B(N) \left( \sum_{D \subseteq B} (-1)^{|B\setminus D|} \mu(D \cup C) \right)
\]

\[
= \sum_{C \subseteq N \setminus B} p_C^{|B|(|N|)} \left( \sum_{D \subseteq B} (-1)^{|B\setminus D|} \mu(D \cup C) \right)
\]

Table 4: The symmetric measure \( \mu_{30} \) and its interaction indices

<table>
<thead>
<tr>
<th>( B )</th>
<th>( (b_1, b_2) )</th>
<th>( \mu_3(B) )</th>
<th>( I_{\text{sh}}^p(B) )</th>
<th>( m_3(B) )</th>
<th>( I_{\text{lp}}^p(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>(0,0)</td>
<td>0</td>
<td>0.5583</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( {1} )</td>
<td>(1,0)</td>
<td>0.30</td>
<td>0.2917</td>
<td>0.3000</td>
<td>0</td>
</tr>
<tr>
<td>( {2} )</td>
<td>(1,0)</td>
<td>0.30</td>
<td>0.2917</td>
<td>0.3000</td>
<td>0</td>
</tr>
<tr>
<td>( {3} )</td>
<td>(0,1)</td>
<td>0.20</td>
<td>0.2083</td>
<td>0.2000</td>
<td>0</td>
</tr>
<tr>
<td>( {4} )</td>
<td>(0,1)</td>
<td>0.20</td>
<td>0.2083</td>
<td>0.2000</td>
<td>0</td>
</tr>
<tr>
<td>( {1,2} )</td>
<td>(2,0)</td>
<td>0.50</td>
<td>-0.3333</td>
<td>-0.1000</td>
<td>-0.2000</td>
</tr>
<tr>
<td>( {1,3} )</td>
<td>(1,1)</td>
<td>0.80</td>
<td>0.0917</td>
<td>0.3000</td>
<td>0.3750</td>
</tr>
<tr>
<td>( {1,4} )</td>
<td>(1,1)</td>
<td>0.80</td>
<td>0.0917</td>
<td>0.3000</td>
<td>0.3750</td>
</tr>
<tr>
<td>( {2,3} )</td>
<td>(1,1)</td>
<td>0.80</td>
<td>0.0917</td>
<td>0.3000</td>
<td>0.3750</td>
</tr>
<tr>
<td>( {2,4} )</td>
<td>(1,1)</td>
<td>0.80</td>
<td>0.0917</td>
<td>0.3000</td>
<td>0.3750</td>
</tr>
<tr>
<td>( {3,4} )</td>
<td>(0,2)</td>
<td>0.30</td>
<td>-0.2833</td>
<td>-0.1000</td>
<td>-0.3333</td>
</tr>
<tr>
<td>( {1,2,3} )</td>
<td>(2,1)</td>
<td>0.90</td>
<td>-0.1500</td>
<td>-0.4000</td>
<td>-0.0741</td>
</tr>
<tr>
<td>( {1,2,4} )</td>
<td>(2,1)</td>
<td>0.90</td>
<td>-0.1500</td>
<td>-0.4000</td>
<td>-0.0741</td>
</tr>
<tr>
<td>( {1,3,4} )</td>
<td>(1,2)</td>
<td>0.85</td>
<td>-0.1000</td>
<td>-0.3500</td>
<td>-0.0196</td>
</tr>
<tr>
<td>( {2,3,4} )</td>
<td>(1,2)</td>
<td>0.85</td>
<td>-0.1000</td>
<td>-0.3500</td>
<td>-0.0196</td>
</tr>
<tr>
<td>( {1,2,3,4} )</td>
<td>(2,2)</td>
<td>1.00</td>
<td>0.5000</td>
<td>0.5000</td>
<td>-0.2143</td>
</tr>
</tbody>
</table>

That is, \( I_{\text{lp}}^p(B) \) depends only on \((b_1, b_2, \ldots, b_p)\).

One can see that Proposition 5 is just a special case of Proposition 8, since the 1-symmetric measure is just the symmetric measure.

**Example 4.** (Adapted from [32]) Let \( N = \{1, 2, 3, 4\}, \mu_3 \) is a symmetric measure on \( N \) with the basis \( \{\{1, 2\}, \{3, 4\}\} \), as given in Table 4.

\[
\mu(d_1 + c_1, d_2 + c_2, \ldots, d_p + c_p).
\]

3.5. The \( k \)-tolerant and \( k \)-intolerant measure

**Definition 12.** (See [30, 29]) Let \( k \in \{1, 2, \ldots, n\} \), a monotone measure \( \mu \) on \( N \) is \( k \)-tolerant if \( \mu(A) = 1 \) for all \( A \subseteq N \) such that \( |A| \geq k \) and there exists a subset \( B \subseteq N \), with \( |B| = k - 1 \), such that \( \mu(B) \neq 1 \). A monotone measure \( \mu \) on \( N \) is \( k \)-intolerant if \( \mu(A) = 0 \).
for all $A \subseteq N$ such that $|A| \leq n - k$ and there exists a subset $B \subseteq N$, with $|B| = n - k + 1$, such that $\mu(B) \neq 0$.

One can see that a $k$-tolerant or $k$-intolerant measure needs $\sum_{i=1}^{k} \binom{n}{i}$ coefficients to be defined [30, 29].

The $k$-tolerant and $k$-intolerant measures describe essentially the tolerance and intolerance character of the Choquet integral [30, 29]. If $\mu$ is $k$-tolerant (resp. $k$-intolerant) measure, then for $x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have $C_{\mu}(x) \geq x(k)$ (resp. $C_{\mu}(x) \leq x(k)$), where $x(k)$ is the $k$th order statistic or $k$th smallest value of $x_1, x_2, \ldots, x_n$ [18, 30, 29]. Especially, the Choquet integral with respect to 1-tolerant (resp. 1-intolerant) measure is just the maximum operator (resp. minimum operator). It also should be pointed out that the 1-tolerant and 1-intolerant measures are respectively the measures $\mu_{en}$ and $\mu_{ep}$ discussed in the literature [40], which describe the extremely positive and negative interaction cases of all the decision criteria, respectively. The monotone measure $\mu_{en}$ is defined as $\mu_{en}(A) = 1$ for $\forall A \subseteq N$ and $A \neq \emptyset$, and $\mu_{en}(\emptyset) = 0$. The monotone measure $\mu_{ep}$ is defined as $\mu_{ep}(N) = 1$ and $\mu_{ep}(A) = 0$ for $\forall A \subseteq N$(i.e., $A \subseteq N$ and $A \neq N$).

If the range of a monotone measure is $\{0, 1\}$, we can call this monotone measure as 0-1 measure. So, the measures $\mu_{en}$ and $\mu_{ep}$ can be called as the 0-1 1-tolerant and 1-intolerant measures, respectively. Similarly, we can get that 0-1 $k$-tolerant measure is just a $k$-tolerant measure such that the measure value of any subset with the cardinality less than $k$ is 0, and the 0-1 $k$-intolerant measure is just a $k$-intolerant measure such that the measure value of any subset with the cardinality larger than $n - k$ is 1.

**Proposition 9.** Let $\mu$ be a 0-1 $k$-tolerant measure on $N$, $A \subseteq N$, then

\[ I_{\text{sum}}^{\mu}(A) = 0 \text{ if } |A| \leq k - 1, \]
\[ I_{\text{sum}}^{\mu}(A) = 1 \text{ if } |A| = k, \]
\[ 0 < I_{\text{sum}}^{\mu}(A) < 1 \text{ if } k + 1 \leq |A| \leq 2k - 2, \]
\[ I_{\text{sum}}^{\mu}(A) = 0 \text{ if } |A| = 2k - 1, \]
\[ I_{\text{sum}}^{\mu}(A) < 0 \text{ if } |A| \geq 2k. \]

**Proof.** For the subset $A \subseteq N$, if $|A| \leq k - 1$, then for $\forall B \subseteq A$, we have $\mu(B) = 0$, and $2^{|A|-1}\mu(A) - \sum_{B \subseteq A} \mu(B) = 0$. That is $I_{\text{sum}}^{\mu}(A) = 0$ if $|A| \leq k - 1$.

For the case $|A| = k$, we have $\mu(A) = 1$, and $\mu(B) = 0$ for $\forall B \subseteq A$, $2^{|A|-1}\mu(A) - \sum_{B \subseteq A} \mu(B) = 2^{|A|-1} - 1$ then $I_{\text{sum}}^{\mu}(A) = 1$.

For the case $k + 1 \leq |A| \leq 2k - 2$, since for $\forall B \subseteq A$ and $|B| \geq k$, we have $\mu(B) = 1$, and then $2^{|A|-1}\mu(A) - \sum_{B \subseteq A} \mu(B) < 2^{|A|-1} - 1$. Furthermore, since

\[ 2^{|A|-1}\mu(A) - \sum_{B \subseteq A} \mu(B) = \frac{1}{2} \left( \sum_{B \subseteq A} (\mu(A) - \mu(B) - \mu(A \setminus B)) \right), \]

for $B \subseteq A$, $\mu(A) - \mu(B) - \mu(A \setminus B) \geq 0$. And there exist at least a subset $B \subseteq A$ such that $\mu(A) - \mu(B) - \mu(A \setminus B) > 0$, then we have $2^{|A|-1}\mu(A) - \sum_{B \subseteq A} \mu(B) > 0$.

For the case $|A| \leq 2k - 1$, since for any $B \subseteq A$, $\mu(A) - \mu(B) - \mu(A \setminus B) = 0$, we have $I_{\text{sum}}^{\mu}(A) = 0$.

For the case $|A| \geq 2k$, for the subsets $B \subseteq A$ such that $k \leq |B| \leq |A| - k$, we have $\mu(A) - \mu(B) - \mu(A \setminus B) = -1$. And for other subsets $B \subseteq A$ such that $|B| < k$ and $|B| > |A| - k$, we have $\mu(A) - \mu(B) - \mu(A \setminus B) = 0$. Hence, we have $I_{\text{sum}}^{\mu}(A) < 0$.

Since 0-1 $k$-intolerant measure is just a 0-1 n-k+1 tolerant measure, by the above proposition, we can have the following proposition.

**Proposition 10.** Let $\mu$ be a 0-1 $k$-intolerant measure on $N$, $A \subseteq N$, then

\[ I_{\text{sum}}^{\mu}(A) = 0 \text{ if } |A| \leq n - k, \]
\[ I_{\text{sum}}^{\mu}(A) = 1 \text{ if } |A| = n - k + 1, \]
\[ 0 < I_{\text{sum}}^{\mu}(A) < 1 \text{ if } n - k + 2 \leq |A| \leq 2n - 2k, \]
\[ I_{\text{sum}}^{\mu}(A) = 0 \text{ if } |A| = 2n - 2k + 1, \]
\[ I_{\text{sum}}^{\mu}(A) < 0 \text{ if } |A| \geq 2n - 2k + 2. \]

For the ordinary $k$-tolerant or $k$-intolerant measure, since the measure value of the nonempty subset is usually not zero, we can only have the following corollaries.

**Corollary 1.** Let $\mu$ be a $k$-tolerant measure on $N$, then $I_{\text{sum}}^{\mu}(A) \leq 0$ if $|A| = 2k - 1$, $I_{\text{sum}}^{\mu}(A) < 0$ for $\forall A \subseteq N$ such that $|A| \geq 2k$.

**Corollary 2.** Let $\mu$ be a $k$-intolerant measure on $N$, then $I_{\text{sum}}^{\mu}(A) = 0$ for $\forall A \subseteq N$ such that $|A| \leq$
\[ n - k, I^u_{\text{sum}}(A) = 1 \text{ if } |A| = n - k + 1, \ I^u_{\text{sum}}(A) < 1 \text{ if } |A| > n - k + 1. \]

4. Conclusions

In this paper, we discussed some properties of the sum interaction index with respect to the particular families of monotone measures, including the \( \lambda \)-measures, the possibility measures, the \( k \)-additive measures, the \( p \)-symmetric measures, and the \( k \)-tolerant and \( k \)-intolerant measures. These results will be useful for the theory and application of the monotone measure based multicriteria decision analysis.

Essentially, the sum interaction index, like the Möbius representation, can be considered as a kind of internal interaction index, which only involves the interactions within the subset. So, in the future, we can further construct the probabilistic sum interaction index, such as Shapley sum interaction index, Banzhaf sum interaction index, to more comprehensively describe the interaction phenomenon among the decision criteria.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (No. 71201110, 71402075, 71502088), the China Postdoctoral Science Foundation funded project (No. 2014M560509, 2015M581907, 2015M581908), the Zhejiang Provincial Natural Science Foundation (No. LY16G 010001), and the K.C. Wong Magna Fund in Ningbo University.

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